ABSTRACT

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<u>OPTIMAL CYLINDRICAL SPINES THAT TRANSFER HEAT BY CONVECTION</u>

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This paper reports the solution of an optimization problem for spines of cylindrical profile that transfer heat by convection when the spine material has constant thermal conductivity. The volume enclosed by the spine is used as a method of rejecting power from the base surface maintained at a specified temperature. The ultimate goal of this thesis is to find a cylindrical spine of minimum profile volumne which transfers a maximum amount of heat by convection when the thermal conductivity is constant. We compute the geometrical and thermal properties of the optimal cylindrical spine.

OPTIMAL CYLINDRICAL SPINES THAT TRANSFER HEAT BY CONVECTION

A THESIS

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CHAPTER 1

ESSENTIAL PRELIMINARIES

1.1. Introduction

In this thesis we conduct a study of cylindrical spines whose profile is selected so that the profile volume (proportional to the mass per unit area of the spine) is a minimum in the class of all spines that transfer a specified power, Q_0, from a base at a specified temperature, T_0. This study is based on a model of the thermal performance of the spine, the assumptions behind which are described in Section 1.2, that imply the possibility of a one-dimensional treatment. We record the basic differential equations and boundary conditions appropriate to this model in Section 1.3, where we also list certain mathematical assumptions (on positivity, continuity, etc.) on the data that are used to justify the subsequent mathematical reasoning.

Much of the analysis of this thesis is valid for an arbitrary mode of heat transfer, $H(T)$, from the spine surface that depends only on the local temperature T. In Section 1.4 we indicate the analytic form of $H(T)$ for convective heat transfer. In this case the surface heat transfer coefficient h can be expressed as $\beta|T - T^*|^\gamma$, in which β and γ are constants.

We next formulate in Section 2.1 a careful statement of the problem for an arbitrary $H(T)$, which is solved in Section 2.2. The results are then restricted to the special case of convective heat transfer when the thermal conductivity $k(T)$ of the spine material is constant. Analysis analogous to that in Section 2.2 for unrestricted spine profiles and arbitrary $H(T)$ is presented in Section 2.3 for cylindrical profiles. In Section 2.4 we discuss cylindrical spines for which the mode of heat transfer is convection and the thermal conductivity is constant.

Section 3.1 contains a listing of the computer program used to find numerical solutions for the problem described in Section 2.4. The numerical results generated from the computer program are discussed and recorded as a table in Section 3.2. We finish with a bibliography of journal articles and books that have been cited in the thesis.

1.2 The basic thin spine assumptions.

Our analysis will be based on several physical assumptions, conventionally utilized in the literature [3] dealing with the one-dimensional treatment of spines. The assumptions are listed as follows.

i) The temperature at any point in the spine is steady, i.e., independent of time.

ii) The temperature does not vary along the cross section of the spine, although it does vary with distance from the base of the spine.

iii) The spine material is homogeneous and isotropic, so that its thermal conductivity at a given point depends only on the temperature at that location.

iv) The temperature of the surrounding environment is uniform.

v) The rate of heat transfer per unit area of spine from a given location on the spine surface depends only on the temperature at that location and the temperature of the surrounding environment.

vi) The temperature and heat flow per unit area of the spine at the base of the spine are uniform, and specified in advance.

vii) There is no impedance to heat transfer from the material supporting the spine to the base of the spine; the temperatures of the supporting material and the base of the spine are the same.

viii) No heat is transferred through the spine tip. This is surely true if the spine tip is insulated, or if the plane of the spine tip is a plane of symmetry between two identical spines that have been manufactured as a single tip to tip unit for structural reasons.

ix) There are no heat sources in the spine.

x) The element ds of arc length along the spine profile can be replaced by the projection dx of that element on the axis of the spine. This requires that the slope of the spine profile be negligible.

1.3. The basic differential equations and boundary conditions.

With reference to Figure 1.1, let $Q(x)$ and $T(x)$ be respectively the heat flow rate and the absolute temperature at the position in the spine where the spine radius is $r(x)$. Then the functions $r(x)$, $T(x)$ and $Q(x)$ satisfy the differential equations,

$$Q(x) = \pi r^2(x) k(T) dT/dx, \quad dQ/dx = 2\pi r(x) H(T), \quad 0 < x < w, \qquad (1.3.1)$$

where the first condition is a consequence of the definition of thermal conductivity and the second condition describes the conservation of energy. In these equations $k(T)$ represents the thermal conductivity of the material and $H(T)$ describes the heat flux through the spine surface. The functions $Q(x)$ and $T(x)$ satisfy the boundary conditions,

$$Q(0) = 0, \quad Q(w) = Q_0, \quad T(w) = T_0, \qquad (1.3.2)$$

in which Q_0 is a specified rate of heat rejection and T_0 is a specified base temperature; the first condition in (1.3.2) describes the basic assumption (viii). In Sections 2.4, 3.1 and 3.2, the mode of heat transfer is restricted to convective heat transfer, and the thermal conductivity, $k(T)$, is assumed to be constant.

In general, however, we require that the data $k(T)$, $H(T)$, Q_0 and T_0 satisfy the following conditions:

A1. $H(T)$ is defined, real and absolutely continuous when $0 \le T$.

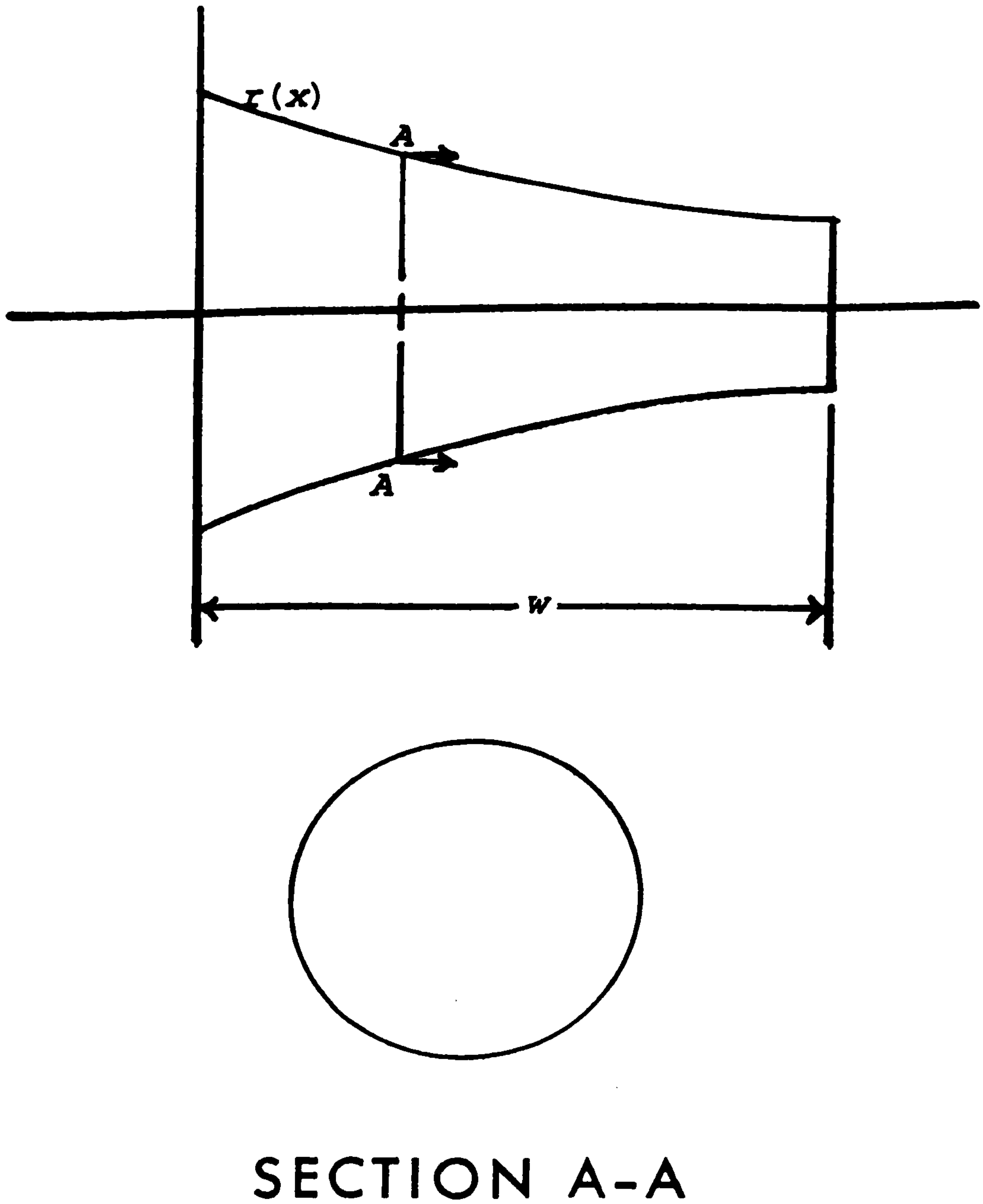

Fig. 1.1. Circular Spine of Unrestricted Profile

A2. There exists a unique temperature T^* such that $0 \le T^* \le T_0$ and $\operatorname{sgn} H(T) = \operatorname{sgn}(T - T^*)$.

A3. There exists a constant M such that $H(T) \le M(T - T^*)$ when $T^* \le T \le T_0$.

A4. $k(T)$ is defined, continuous and bounded below by a positive constant k_1, when $0 \le T \le T_0$. Moreover, $k(T)H(T)$ is integrable over (T^*, T_0).

A5. $Q_0 > 0$, $T_0 > T^*$.

A6. The derivative $H'(T)$ exists when $T^* \le T \le T_0$, and $H'(T)/k(T)H^3(T)$ is a positive, decreasing, absolutely continuous function on every closed interval (T_1, T_0) such that $T^* < T_1 < T_0$.

Each of these conditions will be used when convenient, although they may not be explicitly restated as above.

The effectiveness of the spine, η, is defined as the ratio of the heat actually transferred by the spine to the heat that would be transferred if the entire spine surface were at the temperature T_0, i.e.,

$$\eta = Q_0 \Big/ \left\{ 2\pi H(T_0) \int_0^w r(x)\, dx \right\}. \tag{1.3.3}$$

1.4. Analytic form of $H(T)$.

If the mode of heat transfer from the spine surface is convective, then

$$H(T) = h(T - T^*), \tag{1.4.1}$$

in which h is the surface heat transfer coefficient, which is frequently assumed to be constant, and T^* is the surrounding temperature. On the other hand, empirical relations of the form

$$h = \beta|T - T^*|^\gamma \tag{1.4.2}$$

in which β and γ are positive constants, have been used [2] to correlate heat transfer data in free convection ($\gamma \sim 1/4$), forced convection ($\gamma \sim 1/3$) and nucleate pool boiling ($\gamma \sim 2$). We will use the formula

$$H(T) = 2\beta|T - T^*|^\gamma (T - T^*) \tag{1.4.3}$$

and the phrase "convective heat transfer" as synonyms. When the thermal conductivity $k(T)$ is a positive constant, and $H(T)$ is defined by (1.4.3), all of the conditions A1, A2, A3, A4 and A6 of Section 3 are satisfied.

CHAPTER 2

MATHEMATICAL ANALYSIS OF SPINES

2.1 Statement of the basic optimal spine problem.

The spine volume is

$$V_{pr} = \pi\int_0^w r^2(x)\,dx,$$

because of the basic assumption (x) in Section 1.2 that $ds = dx$. The basic optimum problem is that of finding the spine height w and three continuous funtions $r(x)$, $Q(x)$, and $T(x)$, defined when $0 \le x \le w$, such that $Q(x)$ and $T(x)$ are continuously differentiable when $0 < x < w$, that satisfy the differential equations (1.3.1) when $0 < x < w$, the boundary conditions (1.3.2), and the inequality

$$r(x) > 0 \text{ when } 0 < x \le w, \tag{2.1.1}$$

and for which the spine volume V_{pr} is a minimum.

However, more specific optimum problems arise if the spine profile is subjected to constraints. In particular we will discuss in Section 2.3 the case where $r(x)$ is a constant δ. In this case (see Figure 2.1) the spine is a right circular cylinder with volume $\pi\delta^2 w$. Neither δ nor w is known in advance.

2.2. Solution of the basic optimal spine problem.

In this section we write the differential equations

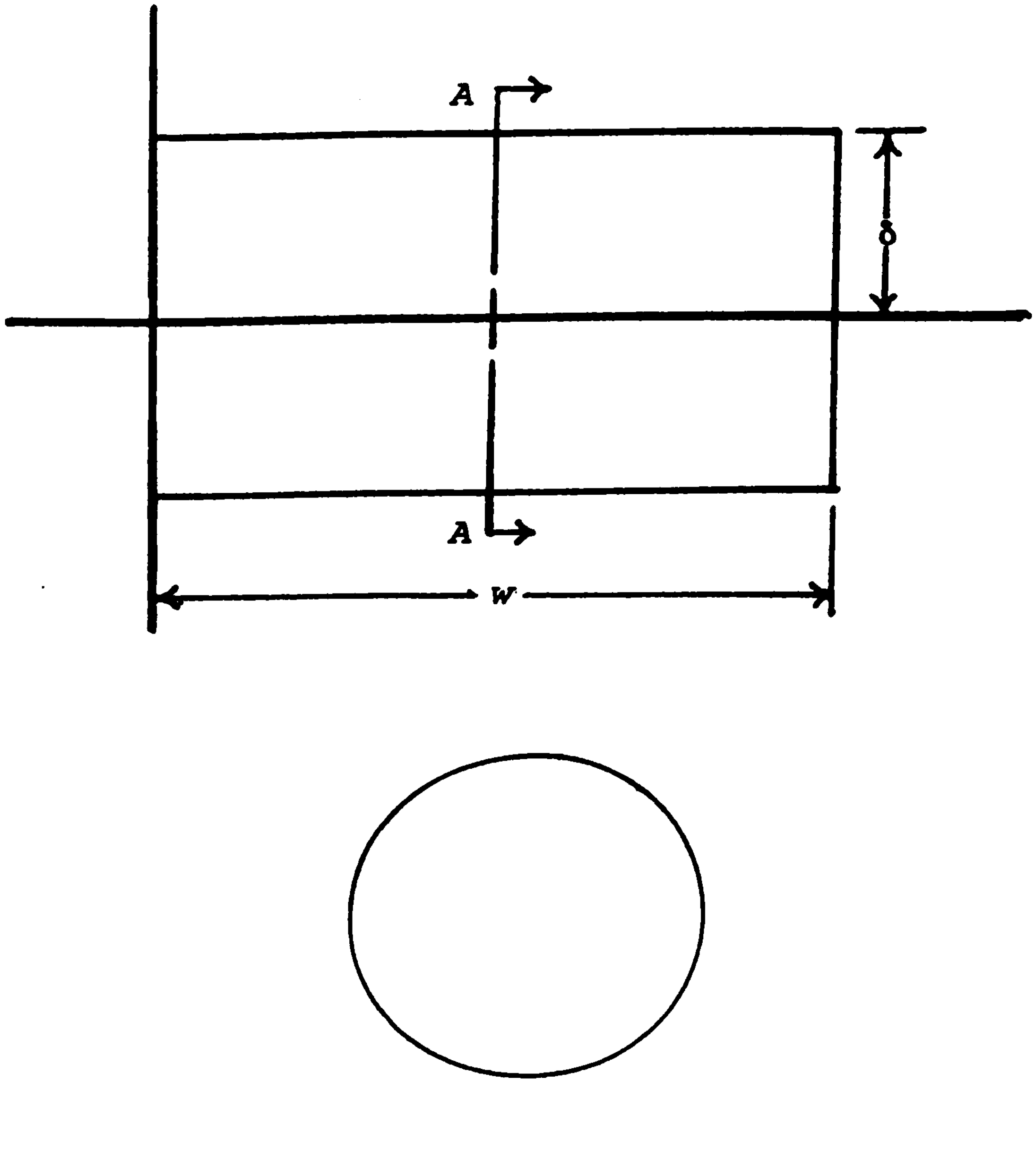

Fig. 2.1. Circular Spine of Cylindrical Profile

(1.3.1) in the form,

$$dxr^{-2}(x) = \pi k(T)\,dT/Q(x)\,,\quad r(x)\,dx = dQ/2\pi H(T)\,.$$

We solve these equations for r and dx by dividing the second differential equation by the first and extracting the cube root. Thus we find that

$$r = \{QdQ/2\pi^2 k(T)\,H(T)\,dT\}^{1/3}, \tag{2.2.1}$$

$$dx = \{k(T)/4\pi H^2(T)\,Q(dQ/dT)\}^{1/3}dQ. \tag{2.2.2}$$

It follows that

$$\pi r^2 dx = \{Q(dQ/du)^4(du/dT)/16\pi^2 k(T)\,H^4(T)\}^{1/3}du.$$

for any function u. If u is defined so that

$$u = U_0^{-1}\int_{T^*}^{T} k(z)\,H^4(z)\,dz, \tag{2.2.3}$$

in which the normalizing constant U_0 is chosen so that $u = 1$ when $T = T_0$, i.e.,

$$U_0 = \int_{T^*}^{T_0} k(z)\,H^4(z)\,dz, \tag{2.2.4}$$

then

$$\pi r^2 dx = \{Q(dQ/du)^4/16\pi^2 U_0\}^{1/3}du.$$

We now define v so that $Q(dQ)^4$ is proportional to $(dv)^4$ and so that $v = 1$ at the spine base where $Q = Q_0$, i.e.,

$$v = (Q/Q_0)^{5/4}. \tag{2.2.5}$$

Then the spine volume is

$$V_{pr} = V^*\int_{u_1}^{1}(dv/du)^{4/3}du, \tag{2.2.6}$$

in which u_1 is the value of u at the spine tip and

$$V^* = \{16Q_0^5/625\pi^2 U_0\}^{1/3}.$$

Moreover, it follows from (2.2.1), (2.2.2), (2.2.3) and (2.2.5) that

$$r = \{2Q_0^2/5\pi^2 U_0\}^{1/3} v^{1/5} H(T)\,dv/du, \tag{2.2.7}$$

$$dx = \{4Q_0U_0/25\pi\}^{1/3}v^{-2/5}H^{-2}(T)\,(dv/du)^{2/3}du. \qquad (2.2.8)$$

We can now restate the basic optimum problem as follows. Find a number u_1 such that $0 \le u_1 < 1$, and a continuous function $v(u)$ defined when $u_1 \le u \le 1$, that is continuously differentiable when $u_1 < u < 1$, that satisfies the boundary conditions

$$v(u_1) = 0, \quad v(1) = 1, \qquad (2.2.9)$$

and the inequality, equivalent to (2.1.1),

$$dv/du > 0 \text{ when } u_1 < u < 1, \qquad (2.2.10)$$

and for which the integral

$$I_{pr} = \int_{u_1}^{1}(dv/du)^{4/3}du \qquad (2.2.11)$$

is a minimum.

It follows from the identity

$$3\zeta^4 = 4\zeta^3 - 1 + (\zeta - 1)^2(3\zeta^2 + 2\zeta + 1),$$

and from (2.2.11) in which $\zeta = (dv/du)^{1/3}$, that

$$3I_{pr} = 3 + u_1 + \int_{u_1}^{1}(\zeta - 1)^2\,\{2\zeta^2 + (\zeta + 1)^2\}du.$$

Therefore, $I_{pr} \ge 1$, i.e.,

$$V_{pr} \ge V^* = (16Q_0^5/625\pi^2U_0)^{1/3}, \qquad (2.2.12)$$

with equality if, and only if, $u_1 = 0$ and $\zeta = 1$, so that $v = u$. In this case, we conclude from (2.2.7), (2.2.5) and (2.2.8) that

$$r = \{2Q_0^2/5\pi^2U_0\}^{1/3}\,u^{1/5}(T)H(T), \quad Q = Q_0u^{4/5}(T), \qquad (2.2.13)$$

$$x = \{4Q_0/25\pi U_0^2\}^{1/3}\int_{T^*}^{T}k(z)H^2(z)\,u^{-2/5}(z)\,dz, \qquad (2.2.14)$$

in which $u(T)$ is defined in (2.2.3). The height of the optimum spine is

$$w = \{4Q_0/25\pi U_0^2\}^{1/3}\int_{T^*}^{T_0}k(z)H^2(z)\,u^{-2/5}(z)\,dz. \qquad (2.2.15)$$

The spine radius r is 0 and the spine temperature T is T^* at the spine tip where $x = 0$, and the spine efficiency is

$$\eta = 5U_0/\left\{4H(T_0)\int_{T^*}^{T_0} k(z)H^3(z)u^{-1/5}zdz\right\}. \tag{2.2.16}$$

The spine radius at the base where $x = w$ is

$$r(w) = \left\{2Q_0^2/5\pi^2 U_0\right\}^{1/3} H(T_0).$$

For the general convective heat transfer case in which $H(T) = \beta T^{\gamma+1}$ (there is no loss of generality in supposing that $T^* = 0$) the above results can be reduced to relatively simple formulas when $k(T)$ is constant. In fact,

$$U_0 = k\beta^4 T_0^{4\gamma+5}/(4\gamma + 5), \quad u = (T/T_0)^{4\gamma+5},$$

$$r = \left\{2Q_0^2(4\gamma + 5)/5\pi^2 k\beta T_0^{\gamma+2}\right\}^{1/3} (T/T_0)^{(9\gamma+10)/5},$$

$$x = \left\{4Q_0 k(4\gamma + 5)^2/(2\gamma + 5)^3\pi\beta^2 T_0^{2\gamma+1}\right\}^{1/3} (T/T_0)^{(2\gamma+5)/5}.$$

Therefore,

$$\begin{aligned}
w &= \left\{20(4\gamma + 5)^2 kQ_0/(2\gamma + 5)^3\pi\beta^2 T_0^{2\gamma+1}\right\}^{1/3},\\
r &= \left\{2(4\gamma + 5)Q_0^2/5\pi^2 k\beta T_0^{\gamma+2}\right\}^{1/3} (x/w)^{(9\gamma+10)/(2\gamma+5)},\\
T &= T_0(x/w)^{5/(2\gamma+5)}, \quad Q = Q_0(x/w)^{4(4\gamma+5)/(2\gamma+5)},\\
\eta &= (11\gamma + 15)/4(4\gamma + 5),\\
V^* &= \left\{16Q_0^5(4\gamma + 5)/625\pi^2 k\beta^4 T_0^{4\gamma+5}\right\}^{1/3}.
\end{aligned} \tag{2.2.17}$$

The simplest special case of (2.2.17) is that in which the surface heat transfer coefficient h is constant. For this case $\gamma = 0$ and $\beta = h$; hence

$$\begin{aligned}
w &= \left\{4kQ_0/\pi h^2 T_0\right\}^{1/3} = 1.08385\left\{kQ_0/h^2 T_0\right\}^{1/3},\\
r &= \left\{2Q_0^2/\pi^2 khT_0^2\right\}^{1/3}(x/w)^2\\
&= 0.587368\left\{Q_0^2/khT_0^2\right\}^{1/3}(x/w)^2,\\
T &= T_0(x/w), \quad Q = Q_0(x/w)^4, \quad \eta = 3/4,\\
V^* &= \left\{16Q_0^5/125\pi^2 kh^4 T_0^5\right\}^{1/3} = 0.234947\left\{Q_0^5/kh^4 T_0^5\right\}^{1/3}.
\end{aligned} \tag{2.2.18}$$

The spine profile is a parabolic arc whose vertex is at the spine tip, and the spine temperature is proportional to x.

2.3. Spines with cylindrical profiles.

In this and subsequent sections we restrict attention to cylindrical spines for which $r(x)$ is a constant δ. We first obtain results for an arbitrary $H(T)$ and then apply these results to the special case of convective heat transfer, i.e., $H(T) = \beta(T - T^*)^{\gamma+1}$, in which β is a positive constant, T^* and γ are nonnegative constants.

The optimal cylindrical spine is determined by numbers δ and w for which the differential equations,

$$Q = \pi\delta^2 k(T)\, dT/dx, \quad dQ/dx = 2\pi\delta H(T), \quad 0 < x < w, \qquad (2.3.1)$$

have a solution $T(x)$, $Q(x)$ that satisfies the boundary conditions

$$Q(0) = 0, \quad Q(w) = Q_0, \quad T(w) = T_0, \qquad (2.3.2)$$

and for which the spine volume, $V_{pr} = \pi\delta^2 w$, is a minimum.

It follows from (2.3.1) that

$$dx/dT = \pi\delta^2 k(T)/Q, \qquad (2.3.3)$$

$$QdQ/dT = 2\pi^2\delta^3 k(T) H(T). \qquad (2.3.4)$$

If we integrate (2.3.4), use the first boundary condition in (2.3.2), and introduce T_1 as the spine tip temperature, we see that

$$Q^2 = 4\pi^2\delta^3 G^2(T_1, T), \qquad (2.3.5)$$

where

$$G(T_1, T) = \left\{\int_{T_1}^{T} k(z) H(z)\, dz\right\}^{1/2}. \qquad (2.3.6)$$

In view of (2.3.2) this implies that

$$\delta = \left[Q_0/\{2\pi G(T_1, T_0)\}\right]^{2/3}. \tag{2.3.7}$$

Moreover, it follows from (2.3.3) and (2.3.5) that

$$x = \delta^{1/2}2^{-1}\int_{T_1}^{T}\left\{k(z)\,dz/G(T_1, z)\right\}. \tag{2.3.8}$$

When $T = T_0$ we infer from (2.3.2), (2.3.8) and (2.3.7) that

$$w = \left[Q_0/\{16\pi G(T_1, T_0)\}\right]^{1/3}\int_{T_1}^{T_0}\{k(T)\,dT/G(T_1, T)\}, \tag{2.3.9}$$

$$V_{pr} = \left[Q_0^5/\{256\pi^2 G^5(T_1, T_0)\}\right]^{1/3}\int_{T_1}^{T_0}\{k(T)\,dT/G(T_1, T)\}. \tag{2.3.10}$$

For each T_1 such that $T^* < T_1 < T_0$ the cylindrical spine with dimensions δ and w determined by (2.3.7) and (2.3.9) has the required thermal performance and a spine volume determined by (2.3.10).

If $T_1 \le T \le T_0$ and T_1 is close to T_0, then it follows from (2.3.6) and (2.3.10) that $G(T_1, T) \sim \{k(T_0)H(T_0)(T - T_1)\}^{1/2}$,

$$\int_{T_1}^{T_0}k(T)\,dT/G(T_1, T) \sim \{k(T_0)/H(T_0)\}^{1/2}\int_{T_1}^{T_0}(T - T_1)^{-1/2}dT$$
$$= 2\{k(T_0)(T_0 - T_1)/H(T_0)\}^{1/2},$$

$$V_{pr} = \{Q_0^5/32\pi^2\}^{1/3}\ \{k(T_0)H^4(T_0)(T_0 - T_1)\}^{-1/3}.$$

Hence, V_{pr} approaches $+\infty$ when T_1 approaches T_0.

Moreover, an integration by parts followed by an appeal to (2.3.6) and the condition A4 shows that

$$\int_{T_1}^{T_0}k(T)\,dT/G(T_1, T) = \int_{T_1}^{T_0}k(T)H(T)\,dT/H(T)G(T_1, T)$$

$$= \int_{T_1}^{T_0}H^{-1}\left\{2\frac{\partial}{\partial T}\ G(T_1, T)\,dT\right\}$$

$$= \{2/H(T)\}G(T_1, T)\Big|_{T_1}^{T_0} + 2\int_{T_1}^{T_0}G(T_1, T)H'(T)\,dT/H^2(T)$$

$$= 2G(T_1, T_0)/H(T_0) + 2\int_{T_1}^{T_0}G(T_1, T)H'(T)\,dT/H^2(T)$$

$$\ge 2\left\{k^{1/2}/H(T_0)\right\}\left\{\int_{T_1}^{T_0}H(z)\,dz\right\}^{1/2}$$

$$+ 2k^{1/2}\int_{T_1}^{T_0}\left\{H'(T)/H^2(T)\right\}\left\{\int_{T_1}^{T}H(z)\,dz\right\}^{1/2}dT \quad (2.3.11)$$

An integration by parts of the second term in (2.3.11) and an appeal to condition A3 shows that the right hand side of (2.3.11) may be written in the form,

$$2k_1^{1/2}H^{-1}(T_0)\left\{\int_{T_1}^{T_0}H(z)\,dz\right\}^{1/2} + 2k_1^{1/2}\left[-\left\{\int_{T_1}^{T}H(z)\,dz\right\}^{1/2}H^{-1}(T)\Big|_{T_1}^{T_0}\right.$$

$$\left. + \int_{T_1}^{T_0}2^{-1}H^{-1}(T)\left\{\int_{T_1}^{T}H(z)\,dz\right\}^{-1/2}H(T)\,dT\right]$$

$$= k_1^{1/2}\int_{T_1}^{T_0}\left\{\int_{T_1}^{T}H(z)\,dz\right\}^{-1/2}dT$$

$$\geq k_1^{1/2}\int_{T_1}^{T_0}M^{-1/2}\left\{\int_{T_1}^{T}(z - T^*)\,dz\right\}^{-1/2}dT$$

$$= (2k_1/M)^{1/2}\int_{T_1}^{T_0}\left\{(T - T^*)^2 - (T_1 - T^*)^2\right\}^{-1/2}dT. \quad (2.3.12)$$

Now let $\cosh u = (T - T^*)/(T_1 - T^*)$. Then observe that (2.3.12) reduces to

$$(2k_1/M)^{1/2}\int_{T_1}^{T_0}du = (2k_1/M)^{1/2}\cosh^{-1}\{(T - T^*)/(T_1 - T^*)\}\Big|_{T_1}^{T_0}$$

$$= (2k_1/M)^{1/2}\cosh^{-1}\left\{(T_0 - T^*)/(T_1 - T^*)\right\},$$

which approaches $+\infty$ as T_1 approaches T^*. Also it follows from A4 that $\lim_{T_1=T^*} G(T_1, T_0) = G(T^*, T_0)$. Hence, we infer from (2.3.10) that V_{pr} approaches $+\infty$ when T_1 approaches T^*. Therefore, the minimum value of V_{pr} is attained at a tip temperature T_1 interior to the interval (T^*, T_0) for which

$$dV_{pr}/dT_1 = 0. \quad (2.3.13)$$

We cannot compute the derivative in (2.3.13) by differentiating the right hand side of (2.3.10) in a straightforward manner, because $G(T_1, T_1) = 0$. However, we can proceed as follows. Because

$$k(T)/G(T_1,T) = 2H^{-1}(T)\frac{\partial}{\partial T}G(T_1,T),$$

an integration by parts in (2.3.10) shows that

$$\left\{2^5\pi^2/Q_0^5\right\}^{1/3}V_{pr} = H^{-1}(T_0)\,G^{-2/3}(T_1,T_0) + G^{-5/3}(T_1,T_0)\int_{T_1}^{T_0}G(T_1,T)\,H'(T)\,H^{-2}(T)\,dT.$$

The right hand side of this equation can be differentiated with respect to T_1, with the result that (2.3.13) holds if, and only if,

$$0 = k(T_1)\,H(T_1)\,2^{-1}G^{-8/3}(T_1,T_0)\Big[2H^{-1}(T_0) + 5G^{-1}(T_1,T_0)\int_{T_1}^{T_0}H'(T)\,H^{-2}(T)\,G(T_1,T)\,dT - 3G(T_1,T_0)\int_{T_1}^{T_0}\frac{H'(T)\,dT}{H^2(T)\,G(T_1,T)}\Big],$$

which implies that the second factor must be equal to zero. Hence, we have

$$0 = 2H^{-1}(T_0) + 5G^{-1}(T_1,T_0)\int_{T_1}^{T_0}H'(T)\,H^{-2}(T)\,G(T_1,T)\,dT - 3G(T_1,T_0)\int_{T_1}^{T_0}H'(T)\,H^{-2}(T)\,G^{-1}(T_1,T)\,dT,$$

$$2H^{-1}(T_0) = \int_{T_1}^{T_0}H'(T)\,H^{-2}(T)\{3G(T_1,T_0)/G(T_1,T) - 5G(T_1,T)/G(T_1,T_0)\}dT$$

$$= \int_{T_1}^{T_0}H'(T)\,H^{-2}(T)\{3P - 5P^{-1}\}dT, \qquad (2.3.14)$$

where $P = G(T_1,T_0)/G(T_1,T)$.

Our earlier analysis guarantees that equation (2.3.14) has at least one solution T_1. If we use condition A6 recorded at the end of Section 1.3, we can show that this solution is unique. In fact an integration by parts in (2.3.14) shows that

$$2H^{-1}(T_0) = H'(T)\,k^{-1}(T)\,H^{-3}(T)\,\Big\{\frac{10}{3}G^3(T_1,T)\,G^{-1}(T_1,T_0)$$

$$- 6G(T_1, T_0)\,G(T_1, T)\Big\}\Big|_{T_1}^{T_0} - \int_{T_1}^{T_0} \Big\{H'(T)\,k^{-1}(T)\,H^{-3}(T)\Big\}'$$

$$\Big\{\frac{10}{3}G^3(T_1, T)\,G^{-1}(T_1, T_0) - 6G(T_1, T_0)\,G(T_1, T)\Big\}dT$$

$$= H'(T_0)\,k^{-1}(T_0)\,H^{-3}(T_0)\ \Big\{\frac{10}{3}G^2(T_1, T_0) - 6G^2(T_1, T_0)\Big\}$$

$$-\int_{T_1}^{T_0} \Big\{H'(T)\,k^{-1}(T)\,H^{-3}(T)\Big\}'\Big\{\frac{10}{3}G^3(T_1, T)\,G^{-1}(T_1, T_0)$$

$$- 6G(T_1, T_0)\,G(T_1, T)\Big\}dT$$

$$= -\frac{8}{3}H'(T_0)\,G^2(T_1, T_0)\,k^{-1}(T_0)\,H^{-3}(T_0)$$

$$- \frac{2}{3}G^2(T_1, T_0)\int_{T_1}^{T_0}\Big\{H'(T)\,k^{-1}(T)\,H^{-3}(T)\Big\}'(5P^{-3} - 9P^{-1})\,dT$$

$$3H^{-1}(T_0) = -G^2(T_1, T_0)\ \Big[\int_{T_1}^{T_0}\Big\{H'(T)\,k^{-1}(T)\,H^{-3}(T)\Big\}'(5P^{-3} - 9P^{-1})\,dT$$

$$+ 4H'(T_0)\,k^{-1}(T_0)\,H^{-3}(T_0)\Big]. \qquad (2.3.15)$$

The derivative with respect to T_1 of the right hand side of (2.3.15) is

$$- k(T_1)\,H(T_1)\ \Big[\int_{T_1}^{T_0}\Big\{H'(T)\,k^{-1}(T)\,H^{-3}(T)\Big\}'(9P^{-1} - 5P^{-3})\,dT$$

$$- 4H'(T_0)\,k^{-1}(T_0)\,H^{-3}(T_0)\Big]$$

$$+ G^2(T_1, T_0)\ \Big[\int_{T_1}^{T_0}\Big\{H'(T)\,k^{-1}(T)\,H^{-3}(T)\Big\}'(9P^{-3} - 15P^{-5})\,dT\ \partial P^2/\partial T_1\Big]$$

$$= 2^{-1}k(T_1)\,H(T_1)\ \Big[\int_{T_1}^{T_0}\Big\{H'(T)\,k^{-1}(T)\,H^{-3}(T)\Big\}'(9P^4 - 8P^3 - 6P^2 + 5)\,P^{-3}dT$$

$$+ 8\Big\{\int_{T_1}^{T_0}\Big\{H'(T)\,k^{-1}(T)\,H^{-3}(T)\Big\}'dT - \Big\{H'(T_0)\,k^{-1}(T_0)\,H^{-3}(T_0)\Big\}\Big\}\Big]$$

$$= 2^{-1}k(T_1)\,H(T_1)\ \Big[\int_{T_1}^{T_0}\Big\{H'(T)\,k^{-1}(T)\,H^{-3}(T)\Big\}'(9P^4 - 8P^3 - 6P^2 + 5)\,P^{-3}dT$$

$$- 8H'(T_1)\,k^{-1}(T_1)\,H^{-3}(T_1)\Big].$$

This derivative is positive, by virtue of condition A6, because

$$9P^4 - 8P^3 - 6P^2 + 5 = (P-1)^2\{4P^2 + 5(P+1)^2\} \geq 0.$$

This result is sufficient to establish the uniqueness of T_1.

2.4 Cylindrical spines and convective heat transfer.

We apply the general theory developed above first to the case of convective heat transfer with a constant heat transfer coefficient h and a constant thermal conductivity k. Then $H(T) = hT$ (when $\gamma = 0$ it is customary to replace β by the surface heat transfer coefficient h) and (2.3.14) reduces to the following equation:

$$2/T_0 = \int_{T_1}^{T_0} T^{-2}\Big[3\Big\{(T_0^2 - T_1^2)/(T^2 - T_1^2)\Big\}^{1/2} - 5\Big\{(T^2 - T_1^2)/(T_0^2 - T_1^2)\Big\}^{1/2}\Big]\,dT.$$

If we introduce the variables ζ and ζ_0 so that $T = T_1 \cosh\zeta$, $T_0 = T_1\cosh\zeta_0$, we find that

$$2\,\mathrm{sech}\,\zeta_0 = \int_0^{\zeta_0} \mathrm{sech}^2\zeta\,(3\sinh\zeta_0 - 5\sinh^2\zeta\,\mathrm{csch}\zeta_0)\,d\zeta$$

$$= 3\sinh\zeta_0\tanh\zeta_0 - 5(\zeta_0 - \tanh\zeta_0)\,\mathrm{csch}\,\zeta_0,$$

$$5\zeta_0 = 3\sinh\zeta_0\cosh\zeta_0,$$

$$(\sinh 2\zeta_0)/2\zeta_0 = 5/3. \qquad (2.4.1)$$

The unique positive solution ζ_0 of (2.4.1) found by Newton's Method is such that $\zeta_0 = 0.9192964$ and $\tanh\zeta_0 = 0.725564$. Therefore, the dimensions, the spine volume, the spine tip temperature,and the spine efficiency of the optimum cylindrical spine are as follows:

$$\delta = \Big\{Q_0^2/2\pi^2 hk(T_0^2 - T_1^2)\Big\}^{1/3} = \Big\{Q_0^2/2\pi hkT_0^2\Big\}^{1/3}\coth^{2/3}\zeta_0$$

$$= 0.458254\,\Big\{Q_0^2/hkT_0^2\Big\}^{1/3},$$

$$w = \left\{Q_0 k/4\pi h^2 (T_0^2 - T_1^2)^{1/2}\right\}^{1/3} \int_{T_1}^{T_0} (T^2 - T_1^2)^{-1/2} dT$$
$$= \left\{ Q_0 k/4\pi h^2 T_0 \right\}^{1/3} \zeta_0 \coth^{1/3} \zeta_0$$
$$= 0.440041 \left\{Q_0 k/h^2 T_0\right\}^{1/3}, \qquad (2.4.2)$$
$$V_{pr} = \pi\delta^2 w = 0.290306 \left\{ Q_0^5/h^4 k T_0^5 \right\}^{1/3},$$
$$T_1 = T_0 \operatorname{sech} \zeta_0 = 0.688154\ T_0,$$
$$\eta = Q_0/2\pi\delta w h T_0 = 0.789261.$$

We observe that the volume, and efficiency, of the optimum cylindrical spine are about 23.6% greater than, and 5.2% less than, the volume, and efficiency, of the (unrestricted profile) optimum spine reported in (2.2.18).

For the general case of convective heat transfer in which $H(T) = \beta T^{\gamma+1}$, equation (2.3.14) holds if and only if

$$2T_0^{-\gamma-1}(T_0^{\gamma+2} - T_1^{\gamma+2}) = (\gamma + 1)\left\{3(T_0^{\gamma+2} - T_1^{\gamma+2}) I - 5K\right\},$$

in which

$$I = \int_{T_1}^{T_0} T^{-\gamma-2} (T^{\gamma+2} - T_1^{\gamma+2})^{-1/2} dT,$$

$$K = \int_{T_1}^{T_0} T^{-\gamma-2} (T^{\gamma+2} - T_1^{\gamma+2})^{1/2} dT.$$

Because

$$T_1^{\gamma+2} \int_{T_1}^{T_0} T^{-\gamma-2} (T^{\gamma+2} - T_1^{\gamma+2})^{-1/2} dT + \int_{T_1}^{T_0} T^{-\gamma-2} (T^{\gamma+2} - T_1^{\gamma+2})^{1/2} dT$$
$$= \int_{T_1}^{T_0} (T^{\gamma+2} - T_1^{\gamma+2})^{-1/2} dT,$$

it is obvious that $IT_1^{\gamma+2} + K = J$, in which

$$J = \int_{T_1}^{T_0} (T^{\gamma+2} - T_1^{\gamma+2})^{-1/2} dT. \qquad (2.4.3)$$

An integration by parts of K shows that

$$2(\gamma + 1)K = -2T_0^{-\gamma-1}(T_0^{\gamma+2} - T_1^{\gamma+2})^{1/2} + (\gamma + 2)J.$$

It follows that

$$2(\gamma + 1)I = 2T_0^{-\gamma-1}T_1^{-\gamma-2}(T_0^{\gamma+2} - T_1^{\gamma+2})^{1/2} + \gamma T_1^{-\gamma-2} J.$$

Hence, (2.3.14) holds if, and only if,

$$J = \frac{6\{T_1^{-\gamma-2}(T_0^{\gamma+2} - T_1^{\gamma+2}) + 1\}\, T_0^{-\gamma-1}(T_0^{\gamma+2} - T_1^{\gamma+2})^{1/2}}{5(\gamma + 2) - 3\gamma(T_0^{\gamma+2}T_1^{-\gamma-2} - 1)}. \tag{2.4.4}$$

Now if we let $t = T_0/T_1$ and $s = T/T_1$ and recall the definition (2.4.3) of J, we find that

$$J = \int_1^t (T_1^{\gamma+2}s^{\gamma+2} - T_1^{\gamma+2})^{-1/2}T_1 ds = T_1^{-\gamma/2}\int_1^t (s^{\gamma+2} - 1)^{-1/2}ds.$$

Thus,

$$J = T_1^{-\gamma/2}N_0(t,\gamma)$$

in which

$$N_0(t,\gamma) = \int_1^t (s^{\gamma+2} - 1)^{-1/2}ds. \tag{2.4.5}$$

Also because $T_0 = tT_1$, (2.4.4) can now be written in the form,

$$J = 6tT_1^{-\gamma/2}(t^{\gamma+2} - 1)^{1/2}/\{5(\gamma + 2) - 3\gamma(t^{\gamma+2} - 1)\}.$$

Therefore,

$$N_0(t,\gamma) = 6t(t^{\gamma+2} - 1)^{1/2}/(8\gamma + 10 - 3\gamma t^{\gamma+2}). \tag{2.4.6}$$

If we introduce the variables y, τ and μ, defined so that

$$s = (1 - y\tau)^{-1/(\gamma+2)}, \quad \tau = 1 - t^{-\gamma-2}, \quad \mu = \gamma/\{2(\gamma + 2)\},$$

into (2.4.5) we see that

$$N_0(t,\gamma) = \int_0^1 \{(1 - y\tau)^{-1} - 1\}^{-1/2} \{(1 - y\tau)^{\frac{-\gamma-3}{\gamma+2}}\tau/(\gamma + 2)\}dy$$

$$= \{\tau^{1/2}/(\gamma + 2)\}\int_0^1 y^{-1/2}(1 - y\tau)^{\mu-1}dy.$$

The integral can be expressed in terms of a hypergeometric function [1;558,15.3.1] so that

$$N_0(t,\gamma) = 2\tau^{1/2}(\gamma + 2)^{-1}F(1 - \mu, 1/2; 3/2; \tau),$$

and (2.4.6) holds if, and only if,

$$6t(t^{\gamma+2} - 1)^{1/2}/(8\gamma + 10 - 3\gamma t^{\gamma+2})$$
$$= 2\tau^{1/2}(\gamma + 2)^{-1}F(1 - \mu, 1/2; 3/2; \tau),$$

or if, and only if,

$$\{5 - (6\mu + 5)\tau\} F(1 - \mu, 1/2; 3/2; \tau) = 3(1 - \tau)^{\mu}. \qquad (2.4.7)$$

Because [1;556,15.1.5] $F(1,1/2;3/2;\tau) = \tau^{-1/2}\tanh^{-1}(\tau^{1/2})$, this equation reduces to (2.4.1) when $\gamma = 0$, if τ is replaced by $\tanh^2\zeta_0$. Therefore, $\tau = 0.52644$ when $\mu = 0$. This value of τ is so large that a direct attempt to solve (2.4.7) for τ when $\mu = 0$, using its power series expansion [1;556,15.1.1] to calculate the hypergeometric function would be quite tedious. Although the situation may be somewhat better for larger values of γ, or μ, we observe that $\mu = 1/2$ when $\gamma = +\infty$, and [1;556,15.1.6] that

$$F(1/2,1/2;3/2;\tau) = \tau^{-1/2}\sin^{-1}(\tau^{1/2})$$

so that (2.4.7) holds if, and only if $(1 + 4\cos\chi)\chi = 3\sin\chi$, in which $\tau = \sin^2(\chi/2)$. Therefore, $\chi = 1.24404$ and $\tau = 0.33951$. Even in this limiting case the value of τ is sufficiently large that the power series expansion of the hypergeometric function in (2.4.7) is not rapidly convergent.

If follows, however, from [1;559,15.3.6;556,15.1.8; 561,15.3.24] that

$$\begin{aligned}
F(1 - \mu, 1/2; 3/2; \tau) &= \{\Gamma(3/2)\Gamma(\mu)/\Gamma(1/2 + \mu)\Gamma(1)\} \\
&\quad \times F(1 - \mu,\ 1/2;\ 1 - \mu;\ 1 - \tau) + (1 - \tau)^{\mu}\,\Gamma(3/2)\Gamma(-\mu) \\
&\quad \times F(1/2 + \mu,\ 1;\ 1 + \mu;\ 1 - \tau)/\Gamma(1 - \mu)\Gamma(1/2) \\
&= \Gamma(1/2)\Gamma(\mu)\tau^{-1/2}/\{2\Gamma(1/2 + \mu)\} \\
&\quad - (2\mu)^{-1}(1 - \tau)^{\mu}F(1/2 + \mu, 1;\ 1 + \mu;\ 1 - \tau) \\
&= \Gamma(1/2)\Gamma(\mu)\tau^{-1/2}/\{2\Gamma(1/2 + \mu)\}
\end{aligned}$$

$$- (2\mu)^{-1}(1 - \tau)^{\mu}\tau^{-1/2}F(2\mu, 1;\ 1 + \mu;\ \nu)$$

$$= (2\mu\tau^{1/2})^{-1}\left[\Gamma(1/2)\Gamma(1 + \mu)/\Gamma(1/2 - \mu)\right.$$

$$\left.- (1 - \tau)^{\mu}F(2\mu,\ 1;\ 1 + \mu;\ \nu)\right],$$

in which $\nu = (1 - \tau^{1/2})/2$, so that $\tau^{1/2} = 1 - 2\nu$, $\tau = 1 - 4\nu(1 - \nu)$.

Therefore, (2.4.7) holds if, and only if

$$\{5 - (6\mu + 5)\tau\}\{\Gamma(1/2)\Gamma(1 + \mu)/\Gamma(1/2 + \mu)$$

$$- (1 - \tau)^{\mu}\ F(2\mu,\ 1;\ 1 + \mu;\ \nu)\} = 6\mu\tau^{1/2}(1 - \tau)^{\mu},$$

$$F(2\mu,\ 1;\ 1 + \mu;\ \nu) = \Gamma(1/2)\Gamma(1 + \mu)/\ \{\Gamma(1/2 + \mu)\ (1 - \tau)^{\mu}\}$$

$$- 6\mu\tau^{1/2}/\{5 - (6\mu + 5)\tau\}.$$

But recall that $\tau = 1 - 4\nu(1 - \nu)$ so that $(1 - \tau)^{\mu} = \{4\nu(1 - \nu)\}^{\mu}$.

Hence,

$$F(2\mu,\ 1;\ 1 + \mu;\ \nu) = \Gamma(1/2)\Gamma(1 + \mu)/\left[\Gamma(1/2 + \mu)\ \{4\nu(1 - \nu)\}^{\mu}\right]$$

$$- 6\mu(1 - 2\nu)/\left[5 - (6\mu + 5)\{1 - 4\nu(1 - \nu)\}\right].$$

Therefore,

$$F(2\mu,\ 1;\ 1 + \mu;\ \nu) = \Gamma(1/2)\Gamma(1 + \mu)/\left[\Gamma(1/2 + \mu)\ \{4\nu(1 - \nu)\}^{\mu}\right]$$

$$- (1 - 2\nu)/\left[4\nu(1 - \nu)\{1 + 5/(6\mu)\} - 1\right]. \qquad (2.4.8)$$

We observe that when $\gamma = 0$ and $\gamma = +\infty$, ν varies from 0.13722 to 0.20866. Consequently, the power series expansion for $F(2\mu,\ 1;\ 1 + \mu;\ \nu)$ is more rapidly convergent than that for $F(1 - \mu,\ 1/2; 3/2;\ \tau)$, and it is more feasible numerically to solve (2.4.8) for ν. This has been done when $\gamma = 1/4, 1/3, 1/2(1/2)6$. The dimensions, profile volume, spine tip temperature and spine effectiveness of the optimum cylindrical spine can then be inferred from (2.3.7), (2.3.8) and (2.3.10), and the various definitions. We find that

$$\delta = \{Q_0^2(\gamma + 2)/4\pi^2\beta k T_0^{\gamma+2}\tau\}^{1/3}, \tag{2.4.9}$$

$$w = \left\{Q_0(\gamma + 2)^2 k/16\pi\beta^2 T_0^{\frac{\gamma+2}{2}}\tau^{1/2}\right\}^{1/3} J$$

$$= \left\{Q_0(\gamma + 2)^2 k/2\pi\beta^2 T_0^{2\gamma+1}\right\}^{1/3} 3/\{5(\gamma + 2) - 2\tau(4\gamma + 5)\}, \tag{2.4.10}$$

$$V_{pr} = \left\{Q_0^5(\gamma + 2)^4/32\pi^2\beta^4 k T_0^{4\gamma+5}\tau\right\}^{1/3} 3/\{5(\gamma + 2) - 2\tau(4\gamma + 5)\}, \tag{2.4.11}$$

$$T_1 = T_0/t = T_0(1 - \tau)^{\frac{1}{\gamma+2}}, \tag{2.4.12}$$

$$\eta = Q_0/2\pi\delta w\beta T_0^{\gamma+1} = \{5(\gamma + 2) - 2\tau(4\gamma + 5)\}/3(\gamma + 2). \tag{2.4.13}$$

CHAPTER 3

NUMERICAL ANALYSIS OF OPTIMAL CYLINDRICAL SPINES

3.1 Computer program.

In this chapter we discuss the numerical findings generated from the mathematical analysis of the previous chapters. This particular section contains the FORTRAN computer program used to find numerical solutions. We first, however, list the notations used in the equations of this thesis and their counterparts used in the FORTRAN program. The following list shows the relations between the two notations.

Chapter Two	Program
γ	GAMMA
$\gamma + 2$	G2
μ	M
τ	T
ν	NU
$\delta(\beta k T_0^{\gamma+2}/Q_0^2)^{1/3}$	B
$w(\beta^2 T_0^{2\gamma+1}/Q_0 k)^{1/3}$	C
$V_{pr}(\beta^4 T_0^{4\gamma+5}/Q_0^5)^{1/3}$	D
T_1/T_0	E
η	F
V_{pr}/V^*	G

FORTRAN Computer Program

```
DOUBLE PRECISION GAMMA, G2,M,T,NU,A,B,C,D,E,F,G,H,P,V,X
DOUBLE PRECISION PI
DATA PI / 3.14159265359 /

DOUBLE PRECISION X1,L,H1,H2,H3

DOUBLE PRECISION R,F1,F2,SIGMA,S,W,X,Y,Z
DOUBLE PRECISION C(26)

DATA(C(N),N = 1,26)/1.0000000000000000, 0.5772156649015329,
$                   -0.6558780715202538,-0.0420026350340952,
$                    0.1665386113822915,-0.0421977345555443,
$                   -0.0096219715278770, 0.0072189432466630,
$                   -0.0011651675918591,-0.0002152416741149,
$                    0.0001280502823882,-0.0000201348547807,
$                   -0.0000012504934821, 0.0000011330272320,
$                   -0.0000002056338417, 0.0000000061160950,
$                    0.0000000050020075,-0.0000000011812746,
$                    0.0000000001043427, 0.0000000000077823,
$                   -0.0000000000036968, 0.0000000000005100,
$                   -0.0000000000000207,-0.0000000000000054,
$                    0.0000000000000014, 0.000000000000001/

L = 0.0
NU = 0.1
DO 100 I = 0,13,1
   IF (I.LT.2) THEN
      GAMMA = 1.0/(4.0 - L)
   ELSE
      GAMMA = (L - 1.0)/2.0
   ENDIF

   G2 = GAMMA + 2.0
    M = GAMMA / (2.0 * G2)
     Y = 1.0 + (5.0/(6.0 * M))
    XY = M - 0.5
     S = 0
     V = 0

DO 200 J = 0,25,1
   S = S * M + (C (26 - J))
   V = V * X1 + (C(26 - J))

   200 CONTINUE

   X = SQRT (PI) * (V/S)

   250 Q = 1.0
       W = 0.0
```

```
      K = 0.0
      R = 0.0

      SIGMA = 4.0* NU * (1.0 - NU)

300   IF(Q,GT. 10**(-10)) THEN
      W = W + Q
      Q = Q* (K + 2.0*M) * NU / (K + 1.0 + M)
      K = K + 1.0
      R = R + K * Q
     GOTO 300
    ENDIF

     F1 = X/(SIGMA**M)
     F2 = (1.0 - 2.0* NU)/(SIGMA* Y - 1.0)
     Z = W - F1 + F2
     H1 = R/NU
     H2 = 4.0*M*(1.0 - 2.0* NU)* F1/SIGMA
     H3 = H1 + H2

     H = H3 - (2.0/ (SIGMA* Y - 1.0))-(4.0*Y*(F2**2.0))

     IF (ABS(Z) .LT. (0.00000001)) THEN
     GOTO 400
    ENDIF

    NU = NU - Z/H
    GOTO 250

400   T = (1.0 -2.0 * NU)**2.0
      P = 3.0/(5.0*G2 - 2.0*T*(4.0* GAMMA + 5.0))
      A = GAMMA
      B = (G2/(4.0*PI*PI*T))**(1.0/3.0)
      C = (((T*G2**2.0*PI))**(1.0/3.0))*P
      D = PI*C*(B**2.0)
      E = (1.0 - T)**(1.0/G2)
      F = 1.0/(P*G2)
      G = (5.0*G2/8.0)*((G2/((0.8*GAMMA + 1.0)*T))
          **(1.0/3.0))*P

      WRITE (6, 150)A,B,C,D,E,F,G
150   FORMAT (F6.3,3X, 6F10.5)

      L = L + 1.0
100   CONTINUE

500   STOP
      END
```

3.2 Discussion of numerical results.

The numerical results reported in Table 1 describe various geometrical characteristics of cylindrical spines that transfer heat by convection, i.e., $H(T) = \beta T^{\gamma+1}$. In Section 3.1 we have listed the FORTRAN computer program used to calculate numerical results when $\gamma = 1/4, 1/3, 1/2(1/2)6$. These values of γ are recorded in the first column of Table 1. For each value of γ the second, third and fourth columns describe the geometric properties of the optimal cylindrical spine defined in (2.4.9), (2.4.10) and (2.4.11). The dimensionless tip temperature described by equation (2.4.12) is in column five, and the efficiency, described by equation (2.4.13) is in column six. We have also furnished in Table 1 the appropriate entries when $\gamma = 0$, deduced from (2.4.2). Observe that in column six, the efficiency of the optimal cylindrical spine is almost constant ranging from 77% to 79%. The last column of Table 1 is the ratio of the volume of the optimal cylindrical spine to the volumn of the optimal spine with unrestricted geometrical profile recorded in (2.2.17). This ratio is also almost constant.

Table 1

Geometrical and Thermal Properties of Cylindrical Spine with Minimum Profile Volume for Convective Heat Transfer: $H(T) = \beta T^{\gamma+1}$.

γ	$\delta(\beta k T_o^{\gamma+2}/Q_o^2)^{1/3}$	$w(\beta^2 T_o^{2\gamma+1}/Q_o k)^{1/3}$	$V_{pr}(\beta^4 k T_o^{4\gamma+5}/Q_o^5)^{1/3}$	T_1/T_o	η	Volume ratio
0.000	0.45825	0.44004	0.29031	0.68815	0.78926	1.23562
0.250	0.48605	0.41752	0.30987	0.73725	0.78426	1.24114
0.333	0.49464	0.41094	0.31588	0.75036	0.78297	1.24258
0.500	0.51100	0.39890	0.32724	0.77304	0.78078	1.24504
1.500	0.59246	0.34733	0.38301	0.85320	0.77343	1.25342
1.000	0.55471	0.36962	0.35730	0.82169	0.77625	1.25018
2.000	0.62594	0.32957	0.40566	0.87525	0.77151	1.25564
2.500	0.65618	0.31495	0.42603	0.89155	0.77011	1.25727
3.000	0.68386	0.30262	0.44462	0.90408	0.76905	1.25850
3.500	0.70947	0.29201	0.46177	0.91402	0.76822	1.25948
4.000	0.73336	0.28275	0.47773	0.92209	0.76755	1.26027
4.500	0.75578	0.27456	0.49269	0.92878	0.76699	1.26092
5.000	0.77695	0.26724	0.50680	0.93441	0.76653	1.26146
5.500	0.79702	0.26064	0.52016	0.93921	0.76614	1.26192
6.000	0.81614	0.25465	0.53286	0.94336	0.76580	1.26232

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